Jour. Ind. Soc. Ag. Statistics
45 (3), 1993 : 331-340

Fixed-size Confidence Region Estimation of a Linear Function of k-multinormal Means

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(Received : August, 1991)

Summary

Sequential procedures are developed to construct confidence region of preassigned width and coverage probability for a linear function of k– multinormal means based on 'trivial' and 'optional' solutions for sample sizes. For the first case, second-order approximations are obtained for the expected sample size and coverage probability of the sequential procedure. For the second case, the sequential procedure is proved to be 'asymptotically efficient and consistent' in Chow - Robbins [47] sense. Moderate sample size performances of the proposed sequential procedures are also studied.

Key Words : Multinormal populations, confidence region, linear function of means, sequential procedures, second-order approximations, asymptotic efficiency and consistency.

Introduction

Sequential procedures to construct fixed-width confidence interval for the difference between two normal means and a linear function of k(≥ 2) normal means have been developed and studied, respectively, by Robbins et al. [6] and Ramkaran et al. [5]. Chaturvedi [2] provided a multivariate extension of the sequential procedure developed by Robbins et al. [6] assuming population covariance matrices differing by unequal and unknown scalar multipliers. However, in all these papers, the sequential procedures based on 'optimal' sample size solutions are considered and are proved to be 'asymptotically efficient and consistent' in Chow - Robbins [4] sense, that is, only first-order asymptotics are obtained in view of Woodroofe [8].

In the present note, consider the problem of constructing fixed-size confidence ellipsoid for a linear function of k-multi-normal means assuming that the population covariance matrices differ by unknown and unequal scalar multipliers. Sequential procedures based on 'trivial', as well as, 'optimal' solutions for sample sizes are considered and their relative advantages and disadvantages are discussed. The set-up of the problem can be described as follows.

Let $\{\underline{X}_{ij}; j = 1, 2, \ldots\}$ be a sequence of i.i.d. r.v.'s from the ith ($i = 1, \ldots, k$) p-variate normal population having unknown mean vector $\underline{\mu}_i$ and covariance matrix $\sigma_i^2 \sum$, where $\sigma_i \in (0, \infty)$ is an unknown scalar and $\sum$ is known pxp positive definite matrix. For preassigned d, α ($d > 0$, $0 < \alpha < 1$) and given non-zero constants λ_i's, our goal is to construct an ellipsoidal confidence region for $\underline{\mu} = \sum_{i=1}^{k} \lambda_i \underline{\mu}_i$ having maximum diameter 2d and confidence coefficient $\geq \alpha$. Without any loss of generality, one can assume that $\lambda_i = 1$ for all $i = 1, \ldots, k$. Given a random sample $\underline{X}_{i1}, \ldots, \underline{X}_{in_i}$ of size n_i (≥ 2) from the ith population, define $\overline{\underline{X}}_i (n_i) = n_i^{-1} \sum_{j=1}^{ni} \underline{X}_{ij}$ and $\hat{\sigma}_i^2 (n_i) = [p(n_i - 1)]^{-1} \sum_{j=1}^{ni} (\underline{X}_{ij} - \overline{\underline{X}}_{i(n_i)})' \sum^{-1} \cdot (\underline{X}_{ij} - \overline{\underline{X}}_{i(n_i)})$. For $n = \sum_{i=1}^{k} n_i$, use $\overline{\underline{W}}_{(n_i)} = \sum_{i=1}^{k} \overline{\underline{X}}_{(n_i)}$ to estimate $\underline{\mu}$ and define the confidence region to be

$$R_n = [\underline{Z} : (\overline{\underline{W}}_n - \underline{Z})' \sum^{-1} (\overline{\underline{W}}_N - \underline{Z}) \leq d^2]. \tag{1.1}$$

Denoting by F(.), the c.d.f. of a Chi-square variate with p degrees of freedom and using the fact that

$$(\overline{\underline{W}}_n - \underline{\mu})' \sum^{-1} (\overline{\underline{W}}_n - \underline{\mu}) \sim (\sum_{i=1}^{k} \frac{\sigma_i^2}{n_i}) \chi^2_{(p)},$$

one can obtain from (1.1), $$P(\underline{\mu} \in R_n) = F(\sum_{i=1}^{k} \frac{\sigma_i^2}{n_i})^{-1} d^2 \tag{1.2}$$

Let 'a^2' be defined by $$F(a^2) = \alpha. \tag{1.3}$$

It follows from (1.2) and (1.3) that in order to achieve $P(\underline{\mu} \in R_n) \geq \alpha$, $n_1, \ldots, n_k$ must satisfy the inequality

$$\sum_{i=1}^{k} \frac{\sigma_i^2}{n_i} \leq (d/a)^2. \tag{1.4}$$

Thus, for known σ_i's, in order to meet the goals, one can choose n_i in the following two manners:

(i) 'Trivially' choose n_i to be the smallest positive integer $\geq n_i^* = k(a/d)^2 \sigma_i^2$.

(ii) One can take n_i to be the smallest positive integer $\geq n_i^o = (a/d)^2 \sigma_i \sum_{i=1}^{k} \sigma_i$, where n_i^o is the 'optimal' solution for n_i, which minimizes the cost of sampling $n = \sum_{i=1}^{k} n_i$ subject to the constraint (1.4).

However, in the absence of any knowledge about some or all σ_i's, no fixed sample size procedure achieves the goals and motivated by the two kinds of solutions for n_i's, consider two cases in the following sections.

2. *Sequential procedure for 'Trivial' solutions*

The stopping time $N_i = N_i(d)$ from the ith population is defined by

$$N_i = \inf \left[n_i \geq m \geq 2 : n_i \geq k(a/d)^2 \hat{\sigma}_i^2 (n_i) \right], \tag{2.1}$$

where m being the initial sample size. For $N = \sum_{i=1}^{k} N_i$, construct the region R_N for $\underline{\mu}$.

Denoting by $G(X_1, \ldots, x_k) = F\left[ka^2 \left(\frac{N_1^*}{X_1}, \ldots, \frac{n_k^*}{X_K} \right)^{-1} \right]$, the coverage probability associated with the sequential procedure (2.1) is

$$P(\underline{\mu} \in R_N) = E\,[G(N_1, \ldots, N_k)]. \tag{2.2}$$

Before proving the main results, we establish some lemmas.

Lemma 1 : N_i is well-defined stopping rule. (2.3)

$$\lim_{d \to 0} N_i = \infty \text{ a.s.} \tag{2.4}$$

$$\lim_{d \to 0} \left(\frac{N_l}{n_l^*} \right) = 1 \text{ a.s.} \tag{2.5}$$

Proof : The proof of (2.3) can be obtained along the lines of that of result (i) in Lemma 10.9.1 of Zacks [9] p. 560), using the fact that $p(n_l - 1) \frac{\hat{\sigma}_l^2 (n_l)}{\sigma_l^2} = \sum_{j=1}^{n_l - 1} Z_j^{(p)}$, with $Z_j^{(P)} \sim \chi^2_{(p)}$ [see, Wang [7] for proof]. Result (2.4) follows from the definition of N_l and result (2.5) follows from the basic inequality

$$k(a/d)^2 \ \hat{\sigma}_l^2 (N_l) \le N_l \le k(a/d)^2 \ \hat{\sigma}_l^2 (N_l) + (m - 1),$$

on using the fact that $\hat{\sigma}_l^2 (N_l) \xrightarrow{a.s.} \sigma_l^2$ as $N_l \longrightarrow \infty$.

Lemma 2 : As $d \longrightarrow 0$, $P\left(N_l \le \frac{n_l^*}{k}\right) = O(d^{p(m-1)})$.

Proof : See, Lemma 1 in Chaturvedi [3].

Lemma 3 : As $d \longrightarrow 0$, $(n_l^*)^{-1/2} \ (N_l - n_l^*) \xrightarrow{L} N(0, 2p^{-1})$.

Proof : The stopping rule (2.1) can be rewritten as

$$N_l = \inf\left[n_l \ge m - 1 : \sum_{j=1}^{n_l} p^{-1} Z_j^{(p)} \le n_l^2 \ (1 + n_l^{-1}) \ (n_l^*)^{-1}\right]. \tag{2.6}$$

Comparing (2.6) with equation (1.1) of Woodroofe [8], obtain in his notations $\alpha = 2$, $\beta = 1$, $\mu = 1$, $\tau^2 = 2p^{-1}$, $\lambda = n_l^*$, $a = p/2$ and $L_0 = 1$. The lemma now follows from a result of Bhattacharya and Mallik [1] that $(n_l^*)^{-1/2}(N_l - n_l^*) \xrightarrow{L} N(0, \beta^2 \tau^2 \mu^{-2})$, as $d \longrightarrow 0$.

Lemma 4 : For all $m > 1 + 2p^{-1}$, $\frac{(N_l - n_l^*)^2}{n_l^*}$ is uniformly integrable.

Proof : See, Theorem 2.3 of Woodroofe [8].

Lemma 5 : For all $\left|W_i - n_i^*\right| \le \left|N_i - n_i^*\right|$, $i = 1, 2, \ldots, k$,

$$\frac{\partial^2}{\partial N_i^2} G(N_1, \ldots, N_k)\Big|_{(W_1, \ldots, W_k)}$$

and $$\left|\frac{\partial^2}{\partial N_i \partial N_j} G(N_1, \ldots, N_k)\right|_{(W_1, \ldots, W_k)}$$

$$i \ne j = 1, 2, \ldots, k$$

are uniformly bounded on $\left\{N_i > \frac{n_i^*}{k},\ i = 1, 2, \ldots, k\right\}$.

Proof : Denoting by f(.), the p.d.f. of a $\chi^2_{(p)}$ r.v., we have

$$\frac{\partial}{\partial N_i} G(N_1, \ldots, N_k) = \left(\frac{n_i^*}{KN_i^2}\right)\left(\frac{n_1^*}{k N_1} + \ldots + \frac{n_k^*}{k N_k}\right)^{-2} f\left\{\left(\frac{n_1^*}{k N_1}, \ldots, \frac{n_k^*}{k N_k}\right)^{-1}\right\}. \qquad (2.7)$$

$$\frac{\partial^2}{\partial N_i^2} G(N_1, \ldots, N_k)\Big|_{(W_1, \ldots, W_k)}$$

$$= \frac{k}{n_i^{*2}}\left(\frac{n_i^*}{W_i}\right)^3\left(\frac{n_1^*}{W_1} + \ldots + \frac{n_k^*}{W_k}\right)^{-2}$$

$$\left[\frac{n_i^*}{W_i}\left(\frac{n_1^*}{W_1} + \ldots + \frac{n_k^*}{W_k}\right)^{-1}\left\{k^{-1}\left(\frac{p}{2} - 1\right)\left(\frac{n_1^*}{W_1} + \ldots + \frac{n_k^*}{W_k}\right) + \frac{3}{2}\right\} - 2\right]$$

$$f\left\{\left(\frac{n_1^*}{kW_1} + \ldots + \frac{n_k^*}{kW_k}\right)^{-1}\right\}. \qquad (2.8)$$

$$\left|\frac{\partial^2}{\partial N_i \partial N_j} G(N_1, \ldots, N_k)\right|_{(W_1, \ldots, W_k)}$$

$$= \frac{k}{n_i^* n_j^*}\left(\frac{n_i^*}{W_i}\right)^2\left(\frac{n_j^*}{W_j}\right)^2\left(\frac{n_1^*}{W_1} + \ldots + \frac{n_k^*}{W_k}\right)^{-3}$$

$$\left[2 + k\left(\frac{n_1^*}{W_1} + \ldots + \frac{n_k^*}{W_k}\right)^{-1} \left\{k^{-1}\left(\frac{p}{2} - 1\right) - \frac{1}{2}\right\}\right]$$

$$f\left\{\left(\frac{n_1^*}{kW_1} + \ldots + \frac{n_k^*}{kW_k}\right)^{-1}\right\}. \tag{2.9}$$

Note that in the event $\left\{N_i > \frac{n_i^*}{k},\ i = 1, \ldots, k\right\}, \frac{W_i}{n_i^*} \leq \frac{(2k-1)}{k}$ and $\frac{n_i^*}{W_i} \leq k$. Thus, denoting by k, any generic constant independent of d, it is concluded from (2.8) and (2.9) that both are bounded by kd^4, and the lemma follows.

Lemma 6 : For $|W_i - n_i^*| \leq |N_i - n_i^*|$ and all $m > 1 + 2p^{-1}$,

$$\frac{(N_i - n_i^*)^2}{n_i^*} \left\{ \left| \frac{\partial^2}{\partial N_i^2} G(N_1, \ldots, N_k) \right|_{(W_1, \ldots, W_k)} \right\}$$

is uniformly integrable on $\{ N_i > \frac{n_i^*}{k}, i = 1, \ldots, k \}$.

Proof : The lemma follows on combining the results of Lemmas 4 and 5. The main result is now stated in the following theorem, which provides second-order approximations for the expected sample size and coverage probability associated with the sequential procedure (2.1).

Theorem 1 : For all $m > 1 + sp^{-1}$, as $d \longrightarrow 0$,

$$E(N_i) = n_i^* + \nu - (1 + 2p^{-1}) + 0(1), \tag{2.10}$$

$$P(\underline{\mu} \in R_N) = \alpha + \frac{K_p}{k}\left[\left\{(\nu - (1 + 2p^{-1})) + \frac{((p+1) - 2k)}{pk}\right\}\right.$$

$$\sum_{i=1}^{k} \frac{1}{n_i^*} + \frac{(p+1)}{2}\{\nu - (1 + 2p^{-1})\}^2$$

$$\sum_{i<j=1}^{k} \frac{1}{n_i^* n_j^*} + 0(d^2), \tag{2.11}$$

where ν is specified and $K_p = e^{-1/2} \big| 2^{(p/2)} \Gamma(p/2)$.

Proof : Result (2.10) follows from Theorem 2.4 of Woodroofe [8]. Expanding $G(N_1, \ldots, N_k)$ about $(n_1^*, \ldots, n_k^*)$ in second-order Taylor's series for k variables, we obtain for $|W_i - n_i^*| \leq |N_i - n_i^*|$,

$$
\begin{aligned}
P(\underline{\mu} \in R_N) &= G(N_1, \ldots, N_k) + \sum_{i=1}^{k} E(N_i - n_i^*) \left\{ \frac{\partial}{\partial N_i} G(N_1, \ldots, N_k) \Big|_{(n_1^*, \ldots, n_k^*)} \right\} \\
&+ (1/2) \sum_{i=1}^{k} E\left[(N_i - n_i^*)^2 \left\{ \frac{\partial^2}{\partial N_i^2} G(N_1, \ldots, N_k) \Big|_{(W_1, \ldots, W_k)} \right\} \right] \\
&+ 2 \sum_{i<j=1}^{k} E\left[(N_i - n_i^*)(N_j - n_j^*) \left\{ \left| \frac{\partial^2}{\partial N_i \, \partial N_j} G(N_1, \ldots, N_k) \right|_{(W_1, \ldots, W_k)} \right\} \right]
\end{aligned}
\tag{2.12}
$$

Utilizing Lemmas 3, 4, 5 and 6, equations (2.7), (2.8), (2.9) and (2.10), the facts that $W_i \xrightarrow{a.s.} n_i^*$ and $W_j \xrightarrow{a.s.} n_j^*$ as $d \longrightarrow 0$, and the independence of N_i and N_j for all $i \neq j$, we obtained from (2.12), for all $m > 1 + 2p^{-1}$, as $d \longrightarrow 0$,

$$
\begin{aligned}
& P\{(\underline{\mu} \in I_N)\, I\,(N_i > n_i^*/k,\ i = 1, \ldots, k)\} \\
&= \alpha + \frac{K_p}{K} \left\{ \nu - (1 + 2p^{-1}) \right\} \sum_{i=1}^{k} \frac{1}{n_i^*} \\
&+ \frac{K_p}{pk} \{(p+1) - 2k\} \sum_{i=1}^{k} \frac{1}{n_i^*} \\
&+ \frac{(p+1)k_p}{k} \{\nu - (1 + 2p^{-1}) + O(1)\}^2 \sum_{i<j=1}^{k} \frac{1}{n_i^* \, n_j^*} .
\end{aligned}
\tag{2.13}
$$

Further more, since $F(.) \leq 1$, we obtain from (2.2) on using Lemma 2, as $d \to 0$,

$$P\left\{(\underline{\mu} \in R_N)\, I\left(N_i \leq \frac{n_i^*}{k}, i = 1, \ldots, k\right)\right\} \leq \prod_{i=1}^{k} P(N_i \leq \frac{n_i^*}{k})$$

$$= 0\,(d^{kp(m-1)})$$

$$= 0\,(d^2), \text{ for all } m > 1 + p^{-1}. \quad (2.14)$$

Result (2.11) now follows on combining (2.13) and (2.14).

3. *Sequential Procedure for 'Optimal' Solutions*

The following sampling scheme is defined in Ramkaran et al. [5]. Start with a sample of size m (≥ 2) from each of the - k populations. If, upto any stage, $N_i = n_i$ observations have been taken from $\{\underline{X}_{ij}\}$, i = , . . ., k, the next observation is taken from this population, if

$$\frac{n_i}{n'_i} \leq \frac{\hat{\sigma}_i (n_i)}{\hat{\sigma}'_i (n'_i)}, \quad i' \neq i = 1, \ldots, k.$$

The stopping time N=N(d) is the smallest positive integer $n \geq km$ such that

$$n_i \geq (a/d)^2\, \hat{\sigma}_i (n_i) \left(\sum_{i=1}^{k} \hat{\sigma}_i (n_i) \right), \quad i = 1, \ldots, k,$$

where $N = \sum_{i=1}^{k} N_i$.

It can be shown that the coverage probability associated with this sequential procedure is same as given at (2.2). Now we state the following theorem which establishes the results that the sequential procedure defined in this section is 'asymptotically efficient and consistent' in Chow-Robbins [4] sense. Since these results are direct generalizations of the results of Chaturvedit [2], omit the details for brevity.

Theorem 2 : For $n^0 = \sum_{l=1}^{k} n_l^0$,

$$\lim_{d \to 0} E\left(\frac{N}{n_0}\right) = 1, \quad \lim_{d \to 0} P(\underline{\mu} \in R_N) = \alpha.$$

Remarks : It is observed that the sequential procedures motivated by the two kinds of solutions for sample sizes have their own advantages. The procedure based on 'optimal' solutions minimizes the cost function but second-order approximations could not be obtained due to unavailability of the results concerning asymptotic distribution of $(n_l^*)^{-1/2}(N_l - n_l^*)$ and uniform integrability of the square of this quantity. On the other hand, for the sequential procedure based on 'trivial' solutions, these approximations have been obtained successfully.

4. *The Moderate Sample Size Performances of the procedure (2.1)*

The table given below presents the results of Monte-Carlo experiments. We fix $m = 5$, $\alpha = .95$, $k = p = 2$, $\sum = I$, $\underline{\mu}'_1 = (1, 2)$, $\underline{\mu}'_2 = (0,0)$, $\sigma_1 = 1$ and $\sigma_2 = 2$. For various values of d, we conducted 1000 trials. We computed the expected sample sizes $\overline{N}_1, \overline{N}_2$, as well as, the coverage probabilities P that the confidence region covers $\underline{\mu}' = (1, 2)$. The procedure behaves quite satisfactorily.

Table. Results of Monte-Carlo experiments

d	n_1^*	n_2^*	$\overline{N}_1$	$\overline{N}_2$	D
0.440	54.21	108.4	52.07	105.52	.930
.050	41.2	82.4	39.18	79.75	.951
.068	23.4	44.8	22.99	40.90	.949
.071	20.6	41.2	17.54	39.22	.937
.082	15.4	30.7	13.99	30.17	.953
.099	10.5	21.0	7.77	19.00	.954
.100	10.3	20.6	10.02	17.98	.941
.104	9.5	19.1	8.11	18.80	.950

ACKNOWLEDGEMENT

The authors are thankful to Mr. S.A. Akbar, Arkansas State Univerversity for his computational assistance. We are also thankful to the referee for his comments.

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